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THE THEORY OF GYROSCOPIC SYSTEMS WITH NON-CONSERVATIVE FORCES†

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Results obtained previously [1, 2], which are applicable to mechanical systems containing non-conservative positional forces, are developed and generalized. The necessary and sufficient conditions are formulated for the transition to a certain matrix equation, the use of which enables one to overcome the difficulties associated with the existence of non-conservative positional structures in the initial equations. The above-mentioned conditions are expressed directly in terms of the matrix coefficients of the initial equation. This technique is used to analyse the exact equations of a four-gyroscope vertical (without using the equations of precessional theory) under the assumption that it is mounted on a base which moves with respect to the Earth. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

A matrix equation of the form

$$J\ddot{x} + (D + HG)\dot{x} + (\Pi + P)x = X(x, \dot{x}) \tag{1.1}$$

is considered, where $x = \mathrm{col}(x_1, \ldots, x_m)$ is a required vector, $J = J^T$, $D = D^T$, $G = -G^T$, $\Pi = \Pi^T$, $P = -P^T$ (the superscript T denotes transposition) are constant $m \times m$ matrices, $X(x, \dot{x})$ is an m-dimensional column vector containing the components of the vectors x and $\dot{x}$ to powers higher than the first power and $H > 0$ is a certain large scalar parameter. The matrices J and D are assumed to be positive definite and the matrices G and P are non-degenerate.

Equation (1.1) describes the perturbed motion of mechanical systems acted upon by dissipative, gyroscopic, potential and non-conservative positional forces. In systems with gyroscopes, J must be understood as the matrix of the total moments of inertia with respect to the corresponding axes.

The substitution

$$x = L\xi \tag{1.2}$$

has been used previously in [1]. This leads to the equation

$$JL\ddot{\xi} + [2J\dot{L} + (D + HG)L]\dot{\xi} + [J\ddot{L} + (D + HG)\dot{L} + (\Pi + P)L]\xi = \Xi \tag{1.3}$$

where Ξ is a column vector containing the components of ξ and $\dot{\xi}$ to powers higher than unity. Satisfaction of the condition

$$D\dot{L} + PL = 0 \tag{1.4}$$

enables one to eliminate the matrix P from Eq. (1.3). When condition (1.4) is taken into account, Eq. (1.3) can be reduced, apart from the non-linear vector Ξ, to the form [1]

$$\ddot{\xi} + Q\dot{\xi} + R\xi = 0 \tag{1.5}$$

where

$$Q = L^{-1}VL, \quad R = L^{-1}WL \tag{1.6}$$

$$V = J^{-1}(D + HG) + 2A, \quad W = A^2 + J^{-1}(\Pi + HGA), \quad A = D^{-1}P^T \tag{1.7}$$

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It follows from representations (1.6) that the matrices V and Q and W and R respectively are related by a similarity transformation.

Turning to Eq. (1.5) and expressions (1.6) and (1.7), we note that, when the relations

$$V = L^{-1}(t)VL(t), \quad W = L^{-1}(t)WL(t), \quad \forall t \geqslant 0 \tag{1.8}$$

or

$$L(t)V = VL(t), \quad L(t)W = WL(t) \tag{1.9}$$

are satisfied, that is, when the matrices $L(t)$ and V and $L(t)$ and W respectively commute, Eq. (1.5) is considerably simplified and has the form

$$\ddot{\xi} + V\dot{\xi} + W\xi = 0 \tag{1.10}$$

which contains V and W as constant matrix coefficients of $\dot{\xi}$ and ξ.

Multiplying Eq. (1.10) on the left by J, we have

$$J\ddot{\xi} + V_1\dot{\xi} + W_1\xi = 0 \ (V_1 = JV, \ W_1 = JW) \tag{1.11}$$

If the matrix W_1 is symmetrical (there are no non-conservative positional structures) and the matrix V_1 depends on dissipative and gyroscopic forces, then direct application of the Thomason–Tait–Chetayev theorems [3, 4] to Eq. (1.11) is permissible.

The formulation of the problem of obtaining the conditions, which are expressed by means of the matrices occurring in the initial equation (1.1), for which relations (1.8) and (1.9) are identically satisfied and lead to Eq. (1.11), is therefore natural.

2. THE CONDITIONS OF REDUCIBILITY TO EQ. (1.11)

We consider condition (1.4), which can be represented in the form of the matrix equation

$$\dot{L} = AL \tag{2.1}$$

where A is defined by (1.7) From this, we have

$$L = e^{At}L(0) \tag{2.2}$$

where the matrix $L(0)$ corresponds to the initial value of t.

When $L(t)$ and $\dot{L}(t)$ are bounded in the interval $[0, \infty)$ and, also, when $|\det L(t)| \geqslant \delta > 0$, the matrix L will be a Lyapunov matrix. Transformation (1.2) then does not change the stability properties of the linear part of Eq. (1.1).

It can be shown that, with the assumptions which have been made concerning the matrices D and P and when $L(0) = E$ (E is the identity matrix), the solution of the Cauchy problem for Eq. (2.1) will be a Lyapunov matrix. In fact, since the matrix D is symmetrical and positive definite, the unique, symmetrical and positive definite matrices $D^{1/2}$ and $D^{-1/2}$ exist [5]. The notation

$$L_1(t) = D^{1/2}L(t)D^{1/2}, \quad P_1 = -D^{-1/2}PD^{-1/2} \tag{2.3}$$

is used. Then, the Cauchy problem for (2.1) takes the form

$$\dot{L}_1(t) = P_1L_1(t), \quad L_1(0) = D \tag{2.4}$$

Here, the matrix P_1 remains skew-symmetrical. Taking relations (2.3) and (2.4) into consideration, we now have

$$L(t) = D^{-1/2}\exp(P_1t)D^{1/2} \tag{2.5}$$

The estimates for the norms of the matrices $L(t)$ and $\dot{L}(t)$ follow from Eq. (2.5):

$$\|L(t)\| \leq \|D^{-1/2}\| \, \|\exp(P_1 t)\| \, \|D^{1/2}\| = \|D^{-1/2}\| \, \|D^{1/2}\|$$
$$\|\dot{L}(t)\| \leq \|D^{-1/2}\| \, \|P_1\| \, \|\exp(P_1 t)\| \, \|D^{1/2}\| = \|D^{-1/2}\| \, \|P_1\| \, \|D^{1/2}\|, \quad \forall t \geq 0 \tag{2.6}$$

The identity

$$\|\exp(P_1 t)\| \equiv 1 \tag{2.7}$$

has been used here.
The correctness of this identity can be verified by making use of the representation of the Euclidean norm $\|A\|$ of a real matrix $A = \|a_{ij}\|$ [6]

$$\|A\| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} = [\mathrm{tr}(AA^T)]^{1/2} \tag{2.8}$$

where (AA^T) is the trace of the matrix AA^T. Assuming $A = \exp(P_1 t)$ in representation (2.8), we arrive at identity (2.7), since $P_1 = -P_1^T$.

Estimates (2.6) confirm the boundedness of $L(t)$ and $\dot{L}(t)$ in the interval $[0, \infty)$. Now, from (2.5), we have

$$\det L(t) = \det D^{-1/2} \det[\exp(P_1 t)] \det D^{1/2} \equiv 1$$

since, according to the Jacobi identity [6],

$$\det[\exp(P_1 t)] = \exp(t \cdot \mathrm{tr}\, P_1) = 1$$

Thus $L(t)$ is a Lyapunov matrix.

Theorem. Suppose P and G are arbitrary, non-degenerate, skew-symmetrical matrices and that J, D and Π are arbitrary symmetrical matrices and, moreover, the matrices J and D are positive definite. Then, to satisfy conditions (1.8) and (1.9) for any $H > 0$, it is necessary and sufficient to satisfy the conditions

$$PJ^{-1}D = DJ^{-1}P, \quad PD^{-1}G = GD^{-1}P, \quad PD^{-1}\Pi = \Pi D^{-1}P \tag{2.9}$$

Proof. The structure of the matrices P, G, J, D and Π, indicated in the formulation, as well as the positiveness of the parameter H, have been specified in Section 1. We assume that the commutation conditions (1.8) are satisfied or, what is the same thing, that (1.9) is satisfied. Using the representation of a matrix exponent, we have the solution of Eq. (2.3) in the form

$$L(t) = \sum_{l=0}^{\infty} (D^{-1}P^T)^l \frac{t^l}{l!} L(0) \tag{2.10}$$

which holds for any finite t. Taking Eq. (2.10) into account, we obtain from relations (1.9) the equalities

$$D^{-1}PV - VD^{-1}P = 0, \quad D^{-1}PW - WD^{-1}P = 0$$

Using formula (1.7) and separating terms containing the large parameter H as a factor, we write the resulting equalities in the form

$$D^{-1}PJ^{-1}D - J^{-1}P + H(D^{-1}PJ^{-1}G - J^{-1}GD^{-1}P) = 0$$
$$D^{-1}PJ^{-1}\Pi - J^{-1}\Pi D^{-1}P - H(D^{-1}PJ^{-1}G - J^{-1}GD^{-1}P)D^{-1}P = 0 \tag{2.11}$$

Expressions (2.11) must hold for any $H > 0$. This is only possible when the following equalities hold

$$D^{-1}PJ^{-1}D = J^{-1}P, \quad D^{-1}PJ^{-1}G = J^{-1}GD^{-1}P, \quad D^{-1}PJ^{-1}\Pi = J^{-1}\Pi D^{-1}P \tag{2.12}$$

Conditions (2.9) are now easily obtained. Multiplying the first of Eqs (2.12) on the left by D, we arrive at the first of conditions (2.9). Next, multiplying the first of Eqs (2.12) on the right by D^{-1}, we obtain $D^{-1}PJ^{-1} = J^{-1}PD^{-1}$. When account is taken of the resulting condition, the second and third of Eqs (2.12) reduce to the second and third conditions of (2.9) respectively. This proves the necessity.

Now, suppose conditions (2.9) are satisfied. Using formulae (1.7) and (2.12), we verify the correctness of the expressions

$$D^{-1}PV - VD^{-1}P = 0, \quad D^{-1}PW - WD^{-1}P = 0 \tag{2.13}$$

Since, according to Eq. (2.10), the matrix L is uniquely connected with $D^{-1}P$, the matrix L will also commute with V and W. Conditions (2.13) then reduce to conditions (1.9).

This proves the sufficiency and, consequently, the correctness of the theorem.

Note that conditions (2.9) are necessary and sufficient only in order that equations (1.5) should reduce to the form (1.11). However, they do not state anything regarding the properties of the matrix W_1. The essential property of this matrix in investigating the stability of Eq. (1.11) is its symmetry. The following assertion can be verified.

Lemma. In order that the matrix W_1 in Eq. (1.11) should be symmetrical, it is sufficient that the first two of conditions (2.9) are satisfied.

Proof. Using expressions (1.7) and the first two conditions of (2.9) and, also, taking into account the fact that the matrix Π is symmetrical by definition, that is, $\Pi = \Pi^T$, we have

$$\begin{aligned} W_1 - W_1^T &= JW - (JW)^T = J(D^{-1}P)^2 - HGD^{-1}P - (PD^{-1})^2 J + HPD^{-1}G = \\ &= (JD^{-1}P)D^{-1}P - PD^{-1}(PD^{-1}J) + H(PD^{-1}G - GD^{-1}P) = \\ &= PD^{-1}JD^{-1}P - PD^{-1}JD^{-1}P = 0 \end{aligned} \tag{2.14}$$

which it was required to prove.

Condition (2.14) confirms the fact that there are no non-conservative structures in the composition of the matrix W_1. It thereby justifies the legitimacy of using the corresponding Thomson–Tait–Chetayev theorem.

When there are no forces with overall dissipation and there are no arbitrary gyroscopic forces occurring in V_1, the positive definiteness of the symmetrical matrix W_1 corresponds to the stability of the (non-asymptotic) trivial solution of Eq. (1.11) under the above-mentioned conditions. In this case, the addition of forces with overall dissipation and arbitrary gyroscopic forces imparts, by the Thomson–Tait–Chetyev theorem, the property of asymptotic stability to Eq. (1.11).

3. A FOUR-GYROSCOPE VERTICAL

In the light of the theory presented in Sections 1 and 2, a more general version (compared with that analysed earlier [1]) of a four-gyroscope vertical, which has been set up on a base which moves with respect to the Earth, is considered below.

The system is a platform set up in gimbals and this platform is stabilized in the horizon by means of four identical gyroscopes with the vertical axes of their housings. The gyroscopes are connected pairwise by antiparallelograms which ensure that the gyroscopes turn in the plane of the platform through equal angles in opposite directions. Each pair of gyroscopes is connected by means of a spring to the internal framework of the gimbals. It is assumed that the centre of mass of the system is located below its geometric centre.

The platform is controlled by a special correction system which works out the corresponding moments with respect to the axes of the platform and the housings of the gyroscopes. A detailed description of a four-gyroscopic vertical as well as the theory within the framework of precessional formulations are discussed in [7]. The stability of one of the versions of a four-rotor, gyro-horizon for the case of a fixed mounting has been investigated using the complete equations (taking inertial terms into account) by the direct Lyapunov method [8].

The case of a moving point of support, which is applicable to the complete equations, is considered below under the assumption that the base circulates at a linear velocity which is constant in magnitude.

In this case, the equations of motion of the system under consideration in the notation employed in [1] have the form [7] (we neglect the insignificant effect of the diurnal rotation of the Earth)

$$\begin{aligned} &J_1\ddot{x}_1 + b_1\dot{x}_1 + 2H\dot{x}_2 + 2H\omega x_3 + s_1 x_2 + Plx_1 = -Pl\frac{v\omega}{g} \\ &J_2\ddot{x}_2 + b_2\dot{x}_2 - 2H\dot{x}_1 + 2H\omega x_4 + cx_2 - s_2 x_1 = 0 \\ &J_2\ddot{x}_3 + b_2\dot{x}_3 + 2H\dot{x}_4 + 2H\omega x_1 + cx_3 - s_2 x_4 = 2H\frac{v}{R} \\ &J_3\ddot{x}_4 + b_3\dot{x}_4 - 2H\dot{x}_3 + 2H\omega x_2 - s_1 x_3 + Plx_4 = 0 \end{aligned} \tag{3.1}$$

where x_1, x_4 are the angles of inclination of the platform from the plane of the horizon, x_2, x_3 are the angles of inclination of each of the pairs of gyroscopes with respect to the vertical axes of their housings, H is the intrinsic angular momentum of the gyroscope, s_1, s_2 are the positive coefficients of proportionality in the moments of the controlling correction, b_1, b_3, b_2 are the coefficients of viscous friction in the axes of the platform and the gyroscopes respectively, c is the coefficient of rigidity of the springs connecting the housing of the gyroscopes with the internal framework and Pl is the pendulum moment of the system.

Equation (3.1) are relative to a reference trihedron $0\xi\eta\zeta$, with origin at the centre of the suspension, associated with the trajectory of motion of the mounting. In this case, v and ω denote the linear velocity of the suspension point relative to the Earth and the angular velocity of circulation.

The equilibrium positions of system (3.1), which are henceforth denoted by $x_n^*(n = 1, \ldots, 4)$, correspond to the velocity deviators of the gyro-vertical and are determined by the equation

$$Mx^* = F \tag{3.2}$$

where

$$x^* = \text{col}(x_1^*, \ldots, x_4^*), \quad M = \begin{Vmatrix} M_1, & 2H\omega E \\ 2H\omega E & M_2 \end{Vmatrix}$$

$$M_1 = \begin{Vmatrix} Pl & s_1 \\ -s_2 & c \end{Vmatrix}, \quad M_2 = \begin{Vmatrix} c, & s_2 \\ -s_1 & Pl \end{Vmatrix}$$

$$F = \text{col}\left(-Pl\frac{v\,\omega}{g},\ 0,\ 2H\frac{v}{R},\ 0\right)$$

Using the substitution $x_s = x_s^* + y_s$, where x_s^* satisfy Eq. (3.2), system (3.1) reduces to a homogeneous system of equations in the variables y_s which is a special case of Eq. (1.1). We have

$$\begin{aligned} &y = \text{col}(y_1, \ldots, y_4), \quad J = \text{diag}(J_1, J_2, J_2, J_3), \quad D = \text{diag}(b_1, b_2, b_2, b_3) \\ &HG = 2H\,\text{diag}(S, S), \quad P = s\,\text{diag}(S, S), \quad S = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \\ &\Pi = \begin{Vmatrix} T_1 & 2H\omega E \\ 2H\omega E & T_2 \end{Vmatrix}, \quad T_1 = \begin{Vmatrix} Pl & m \\ m & c \end{Vmatrix}, \quad T_2 = \begin{Vmatrix} c & -m \\ -m & Pl \end{Vmatrix} \\ &s = \tfrac{1}{2}(s_1 + s_2), \quad m = \tfrac{1}{2}(s_1 - s_2) \end{aligned} \tag{3.3}$$

Turning now to conditions (2.9), we verify that the first of these conditions is satisfied in the case of the problem being considered if

$$b_s = \mu J_s, \quad s = 1, 2, 3 \tag{3.4}$$

where μ is a certain constant.

We assume that the forces acting on the system, which are modelled by the matrix D in (3.3), are solely due to the small resistance of the medium. Then, conditions (3.4) comply with the previously adopted [1] Sommerfeld–Greenhill concept. In this case, $\mu > 0$ can be understood as a small scalar parameter which depends on the properties of the medium.

In this case, the second of conditions (2.9) is satisfied. When relations (3.4) are taken into consideration, from the third condition we arrive at the equalities

$$s_1 = s_2, \quad cJ_1 = PlJ_2, \quad J_1 = J_3 \tag{3.5}$$

The first of Eqs (3.5) specifies that the same correction characteristics are applicable to all of the coordinates x_s [8]. The second equality can be satisfied by the choice of the system parameters and, in particular, it has been used earlier [7] in the case when $c = P\,l$. The second of Eqs (3.5) can also be satisfied by putting $c = 0, Pl = 0$, which corresponds to there being no springs connecting the housings of the gyroscopes to the internal framework of the suspension and, also, to the fact that there is no pendulum effect (in this case, the system is assumed to be astatic). As it applies to the distribution of the masses presented in [8], we have

$$J_1 = J_1' + 2(A' + A_k + B_k), \quad J_2 = 2(A' + A_k), \quad J_3 = J_3' + 2(A' + A_k + B_k) \tag{3.6}$$

where $J_1' = J_3'$ are the equatorial moments of inertia of the frames with respect to the corresponding axes, A_k, B_k are the equatorial and polar moments of inertia of the housing and A' is the equatorial moment of inertia of the rotor. It follows from expressions (3.6) that the third of conditions (3.5) is satisfied, if $J_1' = J_3'$, that is, when the equatorial moments of inertia of the external and internal frames are equal. The equality $J_1' = J_3'$ can be achieved with a specified degree of approximation in the case of ring gimbals.

If the masses of the rotors predominate considerably over the masses of the remaining suspension elements, then terms containing twice the value of the equatorial moment of inertia of the rotor will be the decisive terms on the right-hand sides of expressions (3.6). If only these terms are taken into account, it is generally possible to assume that

$$J_1 = J_2 = J_3 = 2A'$$

to which, by virtue of relations (3.4), the following relations correspond

$$b_1 = b_2 = b_3 = b, \quad Pl = c$$

Combining ourselves to this case for simplicity, we use Eq. (1.1) and formulae (1.7). Taking account of relations (3.4), we will have [1]

$$\begin{gathered} V_1 = D + h\,\mathrm{diag}(S,\ S), \quad W_1 = \left\| c_{jk} \right\|_1^4 (c_{jk} = c_{kj}) \\ h = 2\mu^{-1}(H\mu - s), \quad c_{jj} = c + 2Hb^{-1}s - 2A'b^{-2}s^2 \\ c_{12} = c_{23} = c_{14} = c_{34} = 0, \quad c_{13} = c_{24} = 2H\omega \end{gathered} \tag{3.7}$$

By Eqs (3.7) dissipative and gyroscopic forces occur in the composition of the matrix V_1. Hence, when there are such forces, the positive definiteness of the matrix W_1 imparts the property of asymptotic stability to the trivial solution of Eq. (1.11). Applying Silvester's criterion to the matrix W_1 we obtain the conditions

$$b^2c + 2(bH - A's) > 0, \quad b^2c + 2bH(s - b\omega) - 2A's^2 > 0 \tag{3.8}$$

Inequalities (3.8) contain, as a special case, the stability conditions for the system under consideration, which are restricted by the limits of precessional theory. In order to obtain these conditions in (3.8), it is necessary to neglect terms containing the quantity A' as a factor. Then, when $b \neq 0$, the first of inequalities (3.8) is always satisfied. From the second inequality, we have the condition

$$bc + 2H(s - b\omega) > 0 \tag{3.9}$$

If $c = 0$, the condition $s > b\omega$ is obtained from inequality (3.9), and this condition is identical to the well-known necessary condition for the stability of a four-rotor gyro-horizon with a radial correction [7].

It should be noted that, when $c \neq 0$, condition (3.9) is also satisfied when $s \leqslant b\omega$ if $bc > 2H(b\omega - s)$. Hence, in this case, the existence of an elastic coupling between the gyroscopes and the internal frame in combination with a pendulum moment helps to reinforce the stability.

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